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The realization of unilateral constraints in mechanical systems with kinetic energy decay☆

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Abstract

A method of realizing unilateral constraints by introducing a small parameter into the kinetic energy, such that the kinetic energy of the system degenerates on a certain manifold in configuration space when the parameter is equal to zero, is considered. The behaviour of the system in the neighbourhood of the degeneracy manifold in the multidimensional case when the small parameter approaches zero is investigated. The results obtained are used in the problem of the motion of a double mathematical pendulum when the mass of the point closest to the suspension point is small.

The realization of constraints in the case when the kinetic energy is degenerate on a certain manifold was first considered by Dirac[1,2] for purposes of quantum mechanics; he investigated the behaviour of trajectories on this manifold, which corresponds to the realization of a bilateral constraint. The "generalized Hamiltonians of the Dirac formalism" was used in Ref. 3 in the problem of realizing a unilateral holonomic constraint by elastic forces. The behaviour of the system in the neighbourhood of the degeneracy manifold was investigated in Ref. 4 in the two-dimensional case.

1. Formulation of the problem

Consider the problem of the motion in space of a system of N particles with masses m_i, constrained by scleronomous holonomic ideal constraints. A force with potential energy $V(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ acts on the particles, where $\mathbf{r}_i$ is the radius vector of the i-th particle. We will assume that the system has n degrees of freedom. We will introduce generalized coordinates $\bar{x} = (x_1, \ldots, x_n)$ in a certain region $\Sigma \subset \mathbf{R}^{3N}$ in the configuration space of the system. Then $\mathbf{r}_i = \mathbf{r}_i(\bar{x})(i = 1, \ldots, N)$.

The kinetic energy of the system in generalized coordinates has the form

$$T = \frac{1}{2}\sum_{i,j=1}^{n} c_{ij}\dot{x}_i\dot{x}_j, \quad c_{ij}(\bar{x}) = \sum_{k=1}^{N} m_k\left(\frac{\partial \mathbf{r}_k}{\partial x_i}, \frac{\partial \mathbf{r}_k}{\partial x_j}\right)$$

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We will assume that the masses of one or several particles are small (of the order of ε^2), in which case the kinetic-energy matrix $C(\bar{x}, \varepsilon^2) = (c_{ij})$ depends on the parameter ε. We will assume that this relationship is such that when $\varepsilon = 0$ the determinant of the matrix $C(\bar{x}, 0)$ vanishes on a certain hypersurface $\Lambda \subset \Sigma$, given by the equation

$$f(\bar{x}) = \det C(\bar{x}, 0) = 0$$

We will assume that $df|_\Lambda \neq 0$. Then Λ is a smooth hypersurface and we can choose local generalized coordinates such that $\Lambda = (x_n = 0)$.

We will investigate the behaviour of the trajectories of the system as $\varepsilon \to 0$.

Example. Consider the problem of the motion of a point inside an $(n-1)$-dimensional ellipsoid

$$Q = \left\{\frac{x_1^2}{a_1^2} + \ldots + \frac{x_n^2}{a_n^2} = 1\right\} \subset \mathbf{R}^n$$

Following Birkhoff,[5] the problem can be regarded as the limit for the problem of the motion of a point on the surface of an n-dimensional ellipsoid

$$\tilde{Q} = \left\{\frac{x_1^2}{a_1^2} + \ldots + \frac{x_n^2}{a_n^2} + \frac{x_{n+1}^2}{a_{n+1}^2} = 1\right\} \subset \mathbf{R}^{n+1}$$

in which the semiaxis a_{n+1} is small (of the order of ε).

By replacing the variables $x_{n+1} \to \varepsilon x_{n+1}$ we can fix the surface of the n-dimensional ellipsoid, in which case the kinetic energy will depend on the small parameter ε and takes the form

$$T = \frac{1}{2}(\dot{x}_1^2 + \ldots + \dot{x}_n^2 + \varepsilon^2 \dot{x}_{n+1}^2)$$

Consider the mapping of the projection of the surface of the n-dimensional ellipsoid onto $\mathbf{R}^n$ along the direction x_{n+1}. The set of critical points of the mapping of the projection forms a manifold on the surface of the n-dimensional ellipsoid, the projection of which coincides with the $(n-1)$-dimensional ellipsoid of the initial problem. It can be shown that when $\varepsilon = 0$ the kinetic energy of the system degenerates on this manifold.

We will write the Lagrange function

$$L(\bar{x}, \dot{\bar{x}}, \varepsilon) = T(\bar{x}, \dot{\bar{x}}, \varepsilon^2) - V(\bar{x}, \varepsilon)$$

where $V(\bar{x}, \varepsilon)$ is the potential energy of the system in generalized coordinates. Lagrange's equations then take the form

$$C(\bar{x}, \varepsilon^2)\ddot{\bar{x}} = F(\bar{x}, \dot{\bar{x}}, \varepsilon); \quad F = (F_i), \quad i = 1, \ldots, n \tag{1.1}$$

where $F_i(\bar{x}, \dot{\bar{x}}, \varepsilon)$ is a function which depends on the potential and kinetic energies.

Definition. Suppose $\gamma_\varepsilon(t) = \gamma(t, \varepsilon)$, $t \in [a, b]$ is the solution of Lagrange's equations (1.1) for certain initial conditions $\gamma_\varepsilon(a) = \bar{x}_0$, $\dot{\gamma}_\varepsilon(a) = \dot{\bar{x}}_0$. If, when $\varepsilon \to 0$, the limit γ_0 of the functions γ_ε exists with respect to the norm C^1, i.e. $\lim ||\gamma_e - \gamma_0||_{C^1} = 0$ as $\varepsilon \to 0$, we will call the curve of γ_0 the limit trajectory.

Since, when $\varepsilon = 0$, the determinant of the matrix $C(\bar{x}, 0)$ is non-zero everywhere, apart from the hypersurface Λ then, by virtue of the theorem of the continuous dependence of the solutions of a differential equation on the parameter, the limiting trajectories in the region $\Sigma\backslash\Lambda$ are solutions of the limit Lagrange equations $C(\bar{x}, 0)\ddot{\bar{x}} = F(\bar{x}, \dot{\bar{x}}, 0)$. Hence, to find the limit trajectory in the region $\Sigma\backslash\Lambda$ it is sufficient to solve Lagrange's equations for $\varepsilon = 0$ in the region $\Sigma\backslash\Lambda$.

2. The behaviour of the trajectories of the system in the neighbourhood of the degeneration hypersurface

We will assume that the potential energy of the system is even in x_n, while the kinetic energy is invariant under the transformation $x_n \to -x_n$, and when $\varepsilon = 0$ it is non-degenerate in $\dot{x}$, where $x = (x_1, \ldots, x_{n-1})$. With these assumptions we can identify the regions $\Sigma_+ = \{(x, x_n), x_n \geq 0\} \subset \Sigma$ and $\Sigma_- = \{(x, x_n), x_n \leq 0\} \subset \Sigma$ and consider the motion of a point in the region Σ_+ with boundary Λ.

A non-degenerate replacement of variables[6] $(x, x_n) \to (q, q_n)$, $q_n = x_n$ exists, which reduces the kinetic energy of the system

$$T(\dot{x}, \dot{x}_n, x, x_n, \varepsilon^2) = \frac{1}{2}(\bar{A}(x, x_n^2, \varepsilon^2)\dot{x}, \dot{x}) + (\bar{b}(x, x_n^2, \varepsilon^2), \dot{x})x_n\dot{x}_n + \frac{1}{2}\bar{c}(x, x_n^2, \varepsilon^2)\dot{x}_n^2$$

to the form

$$T(\dot{q}, \dot{q}_n, q, q_n, \varepsilon^2) = \frac{1}{2}(\hat{A}(q, q_n^2, \varepsilon^2)\dot{q}, \dot{q}) + \frac{1}{2}\hat{c}(q, q_n^2, \varepsilon^2)\dot{q}_n^2$$

Since the kinetic energy of the system is degenerate on Λ when $\varepsilon = 0$ and $\det \hat{A}(q, 0, 0) \neq 0$, we have $\hat{c}(q, 0, 0) = 0$. Consequently $\hat{c}(q, q_n^2, 0) = c(q, q_n^2)q_n^2$. We will assume that $c(q, 0) > 0$; the case $c(q, 0) = 0$ corresponds to a higher-order degeneracy.

As a result, the kinetic energy of the system in the new variables has the form

$$T = \frac{1}{2}([A(q, q_n^2) + \varepsilon^2\tilde{A}(q, q_n^2, \varepsilon^2)]\dot{q}, \dot{q}) + \frac{1}{2}(c(q, q_n^2)q_n^2 + \varepsilon^2\tilde{c}(q, q_n^2, \varepsilon^2))\dot{q}_n^2 \tag{2.1}$$

Suppose $V = V(q, q_n^2, \varepsilon)$ is the potential energy in the new coordinates. Introducing the generalized momenta

$$p = \frac{\partial T}{\partial \dot{q}}, \quad p_n = \frac{\partial T}{\partial \dot{q}_n}$$

we can write Hamilton's function

$$H = \frac{1}{2}(Bp, p) + \frac{p_n^2}{2(cq_n^2 + \varepsilon^2 c)} + V + \varepsilon^2 H_1; \quad B = A^{-1}, \quad H_1 = -\frac{1}{2}(A^{-1}\tilde{A}A^{-1}p, p) \tag{2.2}$$

Hamilton's equations have the form

$$\dot{p} = -\frac{1}{2}(B'_q p, p) + \frac{p_n^2(c'_q q_n^2 + \varepsilon^2\tilde{c}'_q)}{2(cq_n^2 + \varepsilon^2\tilde{c})^2} - \frac{\partial V}{\partial q} - \varepsilon^2\frac{\partial H_1}{\partial q}, \quad \dot{q} = Bp + \varepsilon^2\frac{\partial H_1}{\partial p}$$

$$\dot{p}_n = -\frac{1}{2}(B'_{q_n}p, p) + \frac{p_n^2(c'_{q_n}q_n^2 + 2cq_n + \varepsilon^2\tilde{c}'_{q_n})}{2(cq_n^2 + \varepsilon^2\tilde{c})^2} - \frac{\partial V}{\partial q_n} - \varepsilon^2\frac{\partial H_1}{\partial q_n}, \quad \dot{q}_n = \frac{p_n}{cq_n^2 + \varepsilon^2\tilde{c}}$$

We will expand $B(q, q_n^2)$ and $V(q, q_n^2, \varepsilon)$ in powers of q_n

$$\begin{aligned} B(q, q_n^2) &= B_0(q) + \frac{1}{2}B_1(q)q_n^2 + B_2(q, q_n^2)q_n^4 \\ V(q, q_n^2, \varepsilon) &= \upsilon_0(q, \varepsilon) + \frac{1}{2}\upsilon_1(q, \varepsilon)q_n^2 + \upsilon_2(q, q_n^2, \varepsilon)q_n^4 \end{aligned} \tag{2.3}$$

Making the replacement of variables $p_n = \varepsilon^2 z$, $q_n = \varepsilon y$, taking expansion (2.3) into account, we will write Hamilton's equations in the neighbourhood $\Omega_\varepsilon = \{|p_n| \leq \varepsilon^2, |q_n| \leq \varepsilon\}$ of the manifold $\tilde{\Lambda} = \{p_n = 0, q_n = 0\}$ in phase space, introducing the notation

$$N(p, q) = \upsilon_1(q, 0) + \frac{1}{2}(B_1 p, p) \tag{2.4}$$

$$H_0(p, q) = \frac{1}{2}(B_0(q)p, p) + v_0(q, 0) \tag{2.5}$$

$$F(p, q, z, y) = \frac{z^2}{2(cy^2 + \tilde{c})} + \frac{1}{2}N(p, q)y^2 \tag{2.6}$$

We have

$$\begin{aligned} \dot{p} &= -\frac{\partial H_0}{\partial q} + \varepsilon h_1(p, q, z, y, \varepsilon), \quad \dot{q} = \frac{\partial H_0}{\partial p} + \varepsilon h_2(p, q, z, y, \varepsilon) \\ \varepsilon \dot{z} &= -\frac{\partial F}{\partial y} + \varepsilon f_1(p, q, z, y, \varepsilon), \quad \varepsilon \dot{y} = \frac{\partial F}{\partial z} \end{aligned} \tag{2.7}$$

where h_1, h_2 and f_1 are certain functions; since Hamilton's function is even in the variable q_n we obtain that $f_1(p, q, 0, 0, \varepsilon) = 0$.

Remark. We put $\varepsilon = 0$ and consider the limit system in coordinates q, $s = q_n^2/2$. We will show that the function $N(p, q)$ has the mechanical meaning of the normal reaction to the hypersurface $\Lambda = \{s = 0\}$ for motion over this hypersurface.

We recall that if a constraint $f(q, s) = 0$ is imposed on the system, the motions of the system are described by Lagrange's equations with undetermined multiplier λ

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -\frac{\partial V}{\partial q} + \lambda\frac{\partial f}{\partial q}, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{s}} - \frac{\partial T}{\partial s} = -\frac{\partial V}{\partial s} + \lambda\frac{\partial f}{\partial s}$$

while the Lagrange undetermined multiplier λ is proportional to the reaction of the constraint $f(q, s) = 0$.[6] Since the kinetic energy of the system has the form (2.1) and $f(q, s) = s$, Lagrange's equations take the following form when $\varepsilon = 0$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -\frac{\partial V}{\partial q}, \quad c\ddot{s} + c_s'\dot{s} - \frac{1}{2}(A_s'\dot{q}, \dot{q}) - \frac{1}{2}c_s'\dot{s}^2 = -\frac{\partial V}{\partial s} + \lambda \tag{2.8}$$

Using the expansion

$$A(q, 2s) = A_0(q) + A_1(q)s + A_2(q, s)s^2$$

we find

$$A^{-1} = (A_0(E + A_0^{-1}A_1 s + A_0^{-1}A_2 s^2))^{-1} = A_0^{-1} - A_0^{-1}A_1 A_0^{-1}s + A_2's^2$$

Taking expansion (2.3) into account, we obtain $B_1 = -A_0^{-1}A_1A_0^{-1}$. Since $p = A_0\dot{q}$ when $s = 0$, we have

$$(B_1 p, p) = -(A_0^{-1}A_1A_0^{-1}p, p) = -(A_0^{-1}A_1A_0^{-1}A_0\dot{q}, A_0\dot{q}) = -(A_1\dot{q}, \dot{q})$$

Since, for motion on the constraint,

$$\dot{s} = 0, \quad \ddot{s} = 0, \quad \left.\frac{\partial V}{\partial s}\right|_{s=0} = v_1(q, 0)$$

we obtain from the second equation of (2.8)

$$\lambda = -\frac{1}{2}(A_1\dot{q}, \dot{q}) + v_1(q, 0) = \frac{1}{2}(B_1 p, p) + v_1(q, 0) = N(p, q) \tag{2.9}$$

Consequently, the function $N(p, q)$ has the meaning of the normal reaction to the hypersurface Λ for motion over this surface. The normal to the hypersurface $\Lambda = \{s = 0\}$ is directed into the region $s \geq 0$.

We will henceforth call the function $N(p, q)$ the normal reaction of the constraint.

2.1. The case of a positive normal reaction

We will assume that $N(p, q) > 0$ in a certain region $D \subset \tilde{\Lambda}$. Suppose

$$S = D \times G, \quad G = \{(z, y): |z| \le R, |y| \le R\}$$

We will assume that the initial system belongs to the class C^3. Then

$$H_0, F \in C^3(S); \quad h_1, h_2, f_1 \in C^2(S \times [0, \varepsilon_0])$$

We will consider the Cauchy problem for systems of differential equations (2.7) with initial conditions from the region S

$$p(0) = p_0, \quad q(0) = q_0, \quad z(0) = z_0, \quad y(0) = y_0 \tag{2.10}$$

Putting $\varepsilon = 0$ in system (2.7), we obtain the degenerate system

$$p = -\frac{\partial H_0}{\partial q}, \quad \dot{q} = \frac{\partial H_0}{\partial p}, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0 \tag{2.11}$$

Solving system (2.11) with initial conditions $p(0) = p_0$, $q(0) = q_0$, we obtain

$$p = \tilde{p}(t), \quad q = \tilde{q}(t), \quad z = 0, \quad y = 0 \tag{2.12}$$

Proposition. *Suppose $N(\tilde{p}(t), \tilde{q}(t)) > 0$ for any $t \in [0, T]$. Then, constants C, C_1 and C_2, independent of ε, exist such that for any small $\varepsilon > 0$ the solution of problem (2.7), (2.10) exists for any $t \in [0, T]$ and the following estimates hold*

$$|p(t) - \tilde{p}(t)| + |q(t) - \tilde{q}(t)| < C\varepsilon, \quad |z(t)| < C_1, \quad |y(t)| < C_2; \quad t \in [0, T]$$

Proof. Introducing the notation

$$\xi = (p, q), \quad \eta = (z, y), \quad h = (h_1, h_2), \quad h_0(\xi) = JH_{0\xi}, \quad f = (f_1, 0), \quad J = \begin{Vmatrix} 0 & -E \\ E & 0 \end{Vmatrix}$$

where E is the identity matrix of order $n-1$, we can write system (2.7) in the form

$$\frac{d\xi}{dt} = h_0(\xi) + \varepsilon h(\xi, \eta, \varepsilon), \quad \varepsilon\frac{d\eta}{dt} = \begin{Vmatrix} 0 & -E \\ E & 0 \end{Vmatrix} F_\eta(\xi, \eta) + \varepsilon f(\xi, \eta, \varepsilon) \tag{2.13}$$

We recall that $f_1(\xi, 0, \varepsilon) = 0$.

Since $N(\tilde{\xi}(t)) > 0$, the matrix of the second derivative $F_{\eta\eta}(\xi, \eta)$ of the function F of the form (2.6) is positive definite on the solution (2.12).

We will further use a well-known method of proof.[7] We obtain constants $M > 0$ and $\mu > 0$, which are independent of ε, such that in the region $S \times [0, \varepsilon_0]$ the following estimates hold

$$|f(\xi, \eta, \varepsilon)| \le M|\eta|, \quad |h'_{0_\xi}(\xi)| \le M, \quad |h(\xi, 0, 0)| \le M$$

$$|h'_\xi(\xi, \eta, \varepsilon)| \le M, \quad |h'_\eta(\xi, \eta, \varepsilon)| \le M, \quad |h'_\varepsilon(\xi, \eta, \varepsilon)| \le M, \quad |h_0(\xi) + \varepsilon h(\xi, \eta, \varepsilon)| \le M$$

$$\frac{1}{2}\mu|\eta|^2 \le F(\xi, \eta) \le M|\eta|^2, \quad |F_\xi(\xi, \eta)| \le M|\eta|^2, \quad |F_\eta(\xi, \eta)| \le M|\eta|$$

We find

$$\frac{dF}{dt} = F_\eta\frac{d\eta}{dt} + F_\xi\frac{d\xi}{dt} = F_\eta f + F_\xi(h_0(\xi) + \varepsilon h(\xi, \eta, \varepsilon)) \le$$

$$\le M^2|\eta|^2 + M^2|\eta|^2 \le \frac{4M^2}{\mu}F(\xi, \eta)$$

Integrating, we obtain

$$F(t) \le F(0)e(t), \quad e(t) = \exp\left(\frac{4M^2}{\mu}t\right)$$

Since $F(0) \le M|\eta(0)|^2$ we have $F(t) \le M|\eta(0)|^2 e(t)$. Consequently,

$$|\eta(t)| \le \sqrt{\frac{2M}{\mu}}|\eta(0)|e\left(\frac{T}{2}\right)$$

and the solution $\eta(t)$ is bounded in the section $[0, T]$, i.e.

$$|\eta(t)| \le \tilde{C}, \quad |z(t)| \le C_1, \quad |y(t)| \le C_2; \quad \forall t \in [0, T]$$

where C is a positive constant, independent of ε.

We will estimate the difference between the solution $\xi = \xi(t)$ of system (2.13) and the solution $\tilde{\xi}(t)$ of the degenerate system. To do this we will make the replacement $\xi = \xi(t) + \Delta$. After substitution into the first equation of system (2.13), following the well-known approach,[7] we obtain

$$\frac{d\tilde{\xi}}{dt} + \frac{d\Delta}{dt} = h_0(\tilde{\xi} + \Delta) + \varepsilon h(\tilde{\xi} + \Delta, \eta, \varepsilon)$$

$$\frac{d\Delta}{dt} = h'_{0_\xi}(\tilde{\xi} + \theta\Delta)\Delta + \varepsilon h(\tilde{\xi} + \Delta, \eta, \varepsilon), \quad 0 < \theta < 1$$

Taking into account the estimates presented above we have

$$\frac{d|\Delta|}{dt} \le M|\Delta| + \varepsilon M(1 + |\Delta| + |\eta| + \varepsilon)$$

Then

$$|\Delta(t)| \le \varepsilon(\tilde{C} + \varepsilon)\exp(M(1 + \varepsilon)t) \le \varepsilon(\tilde{C} + 1)\exp(2MT) = C\varepsilon, \quad \forall t \in [0, T]$$

Since $p_n = \varepsilon^2 z$, $q_n = \varepsilon y$, we can formulate the following theorem in the initial variables. □

Theorem 1. *Suppose $N(p, q) > 0$ in a certain region D on the manifold $\tilde{\Lambda}$ in phase space. Suppose $\tilde{p}(t)$, $\tilde{q}(t)$ is the solution of the Hamilton equations with Hamiltonian (2.5) with initial conditions*

$$p(0) = p_0, \quad q(0) = q_0 \in D \tag{2.14}$$

and $N(\tilde{p}(t), \tilde{q}(t)) > 0$ for any $t \in [0, T]$. Then positive constants, C, C_1 and C_2, independent of ε, exist such that, for any small $\varepsilon > 0$ and any initial given $p(0)$, $q(0)$, $p_n(0)$, $q_n(0)$, satisfying conditions (2.14) and the inequalities

$$|p_n(0)| < \varepsilon^2, \quad |q_n(0)| < \varepsilon$$

a solution of Hamilton's equations with Hamiltonian (2.2) exists in the section $[0, T]$ and the following estimates hold

$$|p(t) - \tilde{p}(t)| + |q(t) - \tilde{q}(t)| < C\varepsilon, \quad |p_n(t)| < C_1\varepsilon^2, \quad |q_n(t)| < C_2\varepsilon; \quad \forall t \in [0, T]$$

2.2. The case of a negative normal reaction

We will assume that $N(p, q) < 0$ in a certain region $D \subset \tilde{\Lambda}$. Suppose

$$S = D \times G, \quad G = \{(z, y) : |z| \le R, |y| \le R\}$$

We will assume that the initial system belongs to the class C^∞. Then

$$H_0, F \in C^\infty(S); \quad h_1, h_2, f_1 \in C^\infty(S \times [0, \varepsilon_0]$$

Substituting the solution $p = \tilde{p}(t)$ and $q = \tilde{q}(t)$ of degenerate system (2.11) with initial conditions $(p_0, q_0) \in D$ into the last two equations of system (2.7), we obtain

$$\varepsilon\dot{z} = -\frac{\partial F}{\partial y} + \varepsilon f_1(\tilde{p}, \tilde{q}, z, y, \varepsilon), \quad \varepsilon\dot{y} = \frac{\partial F}{\partial z} \tag{2.15}$$

where

$$F(z, y, t) = F(p, q, z, y)|_{p, q \to \tilde{p}, \tilde{q}} = \frac{z^2}{2(cy^2 + \tilde{c})} + \frac{1}{2}N(\tilde{p}(t), \tilde{q}(t))y^2 \tag{2.16}$$

We put $t = \varepsilon\tau$. Then system (2.15) takes the form

$$\frac{d\eta}{d\tau} = JF_\eta(\eta, t) + \varepsilon f(\eta, t, \varepsilon) \tag{2.17}$$

where

$$\eta = (z, y), \quad J = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}, \quad f = (f_1, 0)$$

Fixing t, we construct the phase portrait of (2.17) when $\varepsilon = 0$. The phase trajectories are specified by the equation $F = \text{const}$. Since $N(\tilde{p}(t), \tilde{q}(t)) < 0$ in the section $[0, T]$, the phase trajectories have the form shown in Fig. 1.

Suppose W_t^s is the stable separatrice of the saddle equilibrium position $z = 0$, $y = 0$, corresponding to the value of the parameter t, and W_t^u is an unstable separatrice. Since the phase trajectories are not closed, for almost all initial conditions the solution of the Cauchy problem for system (2.15) increases without limit as $\varepsilon \to 0$. Hence, almost all the solutions of system (2.7) with initial conditions (2.14) and $z(0) = z_0 \neq 0$, $y(0) = y_0$ depart from the manifold $\tilde{\Lambda}$.

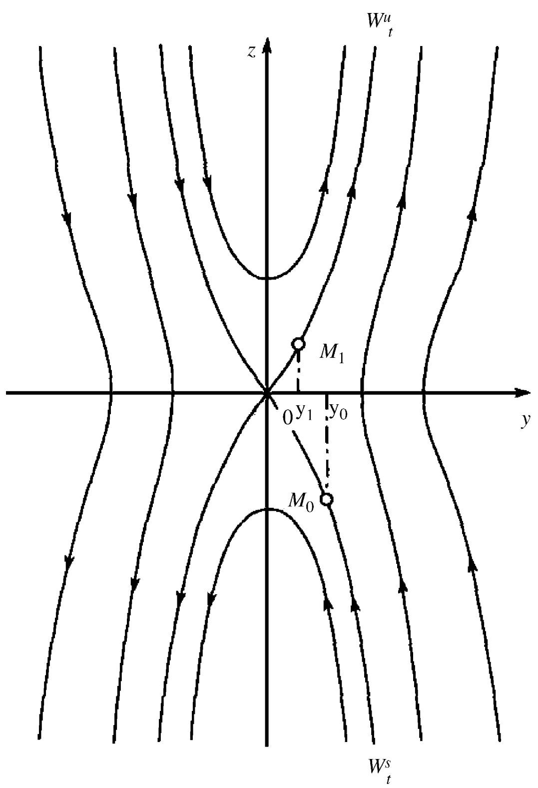

Fig. 1.

Consider the system of differential equations (2.7) with boundary conditions from the region S

$$p(0) = p_0, \quad q(0) = q_0, \quad y(0) = y_0, \quad y(T) = y_1 \tag{2.18}$$

We will seek a solution of problem (2.7), (2.18) in the form[8,9]

$$\zeta(t, \varepsilon) = \chi(t, \varepsilon) + \rho(t/\varepsilon, \varepsilon) + \sigma((t-T)/\varepsilon, \varepsilon)$$

where $\zeta = (p, q, z, y)$, and χ, ρ and σ are regular functions of its arguments. The function $\rho(t/\varepsilon, \varepsilon)$ describes the behaviour of the system in a boundary layer in the neighbourhood of $t = 0$, the function $\sigma((t-T)/\varepsilon, \varepsilon)$ describes the behaviour of the system in the boundary layer in the neighbourhood of $t = T$, while the function $\chi(t, \varepsilon)$ describes the behaviour of the system outside the boundary layers. The solution of the boundary-value problem will be sought in the form of an asymptotic expansion in ε separately for the components of the motion outside and inside the boundary layers. An algorithm for constructing the solution can be found in Ref. 9.

Following this algorithm, we construct the zeroth approximation of the solution of problem (2.7), (2.18). For the variables p and q we have

$$p(t) = \tilde{p}(t) + O(\varepsilon), \quad q(t) = \tilde{q}(t) + O(\varepsilon) \tag{2.19}$$

i.e. the solution $p(t)$ and $q(t)$ in the zeroth approximation in ε does not contain boundary-layer functions and is the solution of degenerate system (2.11). To construct the solution $z(t)$, $y(t)$ we consider system (2.17). Suppose $g_t^\tau : \mathbf{R}^2 \to \mathbf{R}^2$ is the mapping after a time τ, specified by system (2.17), corresponding to the value of the parameter t. On the separatrice W_0^s we take a point M_0 such that $y_{M_0} = y_0$, while on the separatrice W_T^u we take a point M_1 such that $y_{M_1} = y_1$ (Fig. 1). Using well-known results[9] (see also Ref. 8), it can be shown that the solution of system (2.7) for the variables $\eta = (z, y)$ has the form

$$\eta(t) = g_0^{t/\varepsilon}(M_0) + g_T^{(t-T)/\varepsilon}(M_1) + \tilde{\eta}(t) \tag{2.20}$$

where $|\tilde{\eta}| < L\varepsilon$, and L is a constant independent of ε and t. Everywhere, with the exception of a certain neighbourhood of the point $t = 0$, the quantity $g_0^{t/\varepsilon}(M_0)$ can be as small as desired; hence $g_0^{t/\varepsilon}(M_0)$ is the solution in the boundary layer in the neighbourhood of the point $t = 0$. Similarly $g_T^{(t-T)/\varepsilon}(M_1)$ is the solution in the boundary layer in the neighbourhood of $t = T$. Since $M_0 \in W_0^s$ and $M_1 \in W_T^u$, we have, for any instant $t \in [\delta, T-\delta]$ $\delta > 0$,

$$\lim_{\varepsilon \to 0} \eta(t) = g_0^{\infty}(M_0) + g_T^{-\infty}(M_1) = 0$$

uniformly over $t \in [\delta, T-\delta]$. Hence, the solution (2.20) as $\varepsilon \to 0$ approaches the solution of the degenerate system $\eta = 0$.

It follows from well-known results[9] that the solution (2.19), (2.20) is the zeroth approximation of the solution of problem (2.7), (2.18) and the following limits hold

$$|p(t) - \tilde{p}(t)| + |q(t) + \tilde{q}(t)| < C\varepsilon, \quad \left|\eta(t) - g_0^{t/\varepsilon}(M_0) - g_T^{(t-T)/\varepsilon}(M_1)\right| < C_1\varepsilon; \quad \forall t \in [0, T]$$

We can formulate the following theory in the original variables.

Theorem 2. *Suppose $N(p, q) < 0$ in a certain region D on the manifold $\tilde{\Lambda}$ in phase space. Suppose $\tilde{p}(t)$, $\tilde{q}(t)$ is the solution of Hamilton's equations with Hamiltonian (2.5) with initial conditions (2.14) and $N(\tilde{p}(t), \tilde{q}(t)) < 0$ for any $t \in [0, T]$. Then positive constants C, C_1 and C_2, independent of ε, exist such that for any small $\varepsilon > 0$ and any boundary data $p(0)$, $q(0)$, $q_n(0)$, $q_n(T)$, which satisfy conditions (2.14) and the inequalities*

$$|q_n(0)| \le \varepsilon, \quad |q_n(T)| \le \varepsilon$$

a unique solution of Hamilton's equations exist with Hamiltonian (2.2) in the section $[0, T]$ and the following limits hold

$$|p(t) - \tilde{p}(t)| + |q(t) - \tilde{q}(t)| < C\varepsilon, \quad \forall t \in [0, T]$$

$$|p_n(t)| < C_1\varepsilon^3, \quad |q_n(t)| < C_2\varepsilon^2; \quad \forall t \in (0, T)$$

2.3. Interpretation of the results

The results obtained can be interpreted as follows.

1°. Suppose the normal reaction to the hypersurface Λ is positive in a certain region D of the manifold $\tilde{\Lambda}$. Then, for sufficiently small ε, trajectories beginning in the neighbourhood of Ω_ε of the region D in phase space do not leave this neighbourhood during a certain time interval $t \in [0, T]$.

Suppose $\tilde{p}(t)$, $\tilde{q}(t)$ is the solution of the system with constraints $q_n = 0$ for initial given p_0, $q_0 \in D$, and the normal reaction of the constraint, corresponding to the solution $\tilde{p}(t)$, $\tilde{q}(t)$, is positive in the section $t \in [0, T]$. Consider the ε-tube of the solution $\tilde{p}(t)$, $\tilde{q}(t)$ in the phase space

$$\tilde{\Sigma}_\varepsilon = \{(p, p_n, q, q_n) : |p - \tilde{p}(t)| < \varepsilon, \quad |q - \tilde{q}(t)| < \varepsilon, \quad |p_n| < \varepsilon^2, \quad |q_n| < \varepsilon\}$$

For sufficiently small ε the trajectories of the system being investigated, beginning in $\tilde{\Sigma}_\varepsilon$, do not leave the tube

$$\tilde{\Sigma}_{C\varepsilon} = \{(p, p_n, q, q_n) : |p - \tilde{p}(t)| < C\varepsilon, \quad |q - \tilde{q}(t)| < C\varepsilon, \quad |p_n| < C\varepsilon^2, \quad |q_n| < C\varepsilon\}$$

where $C > 0$ is a constant independent of ε, and approaches the solution of the system with the constraint as $\varepsilon \to 0$.

2°. Suppose the normal reaction is negative in a certain region D of the manifold $\tilde{\Lambda}$. Then almost all the trajectories leave the neighbourhood Ω_ε of the region D in phase space in a time of the order $\varepsilon \ln \varepsilon^{-1}$.

Suppose $\tilde{q}(t)$ is the solution of the system with constraint $q_n = 0$ with initial given q_0, $\dot{q}_0$, and the normal reaction of the constraint, corresponding to the solution $\tilde{q}(t)$, is negative in the section $t \in [0, T]$. Consider the ε-tube of the solution $\tilde{q}(t)$ in the configuration space

$$\Sigma_\varepsilon = \{(q, q_n) : |q - \tilde{q}(t)| < \varepsilon, \quad |q_n| < \varepsilon\}$$

For sufficiently small ε, a unique trajectory exists connecting any two points of Σ_ε, which does not leave the tube

$$\Sigma_{C\varepsilon} = \{(q, q_n) : |q - \tilde{q}(t)| < C\varepsilon, \quad |q_n| < C\varepsilon\}$$

and which passes along Λ (Fig. 2). When $\varepsilon \to 0$ this trajectory approaches the trajectory of the system with the constraint.

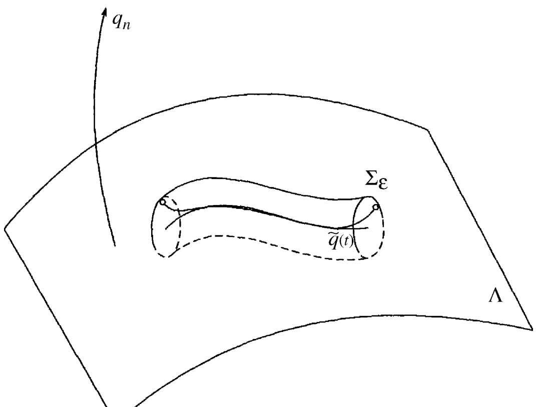

Fig. 2.

2.4. The problem of relaxing the invariance conditions

The invariance conditions with respect to the transformation $x_n \to -x_n$, imposed on the kinetic and potential energies of the system are fairly restrictive.

We will assume that the kinetic energy of the system when $\varepsilon = 0$ degenerates on the surface $\Lambda = \{x_n = 0\}$ and has the form

$$T = \frac{1}{2}(A(x, x_n, \varepsilon^2)\dot{x}, \dot{x}) + (b(x, x_n, \varepsilon^2), \dot{x})\dot{x}_n + \frac{1}{2}c(x, x_n, \varepsilon^2)\dot{x}_n^2, \;\; A = (a_{ij}), \;\; i, j = 1, \dots, n-1$$

Making the change of variables $(x, x_n) \to (q, q_n)$ in accordance with the formulae[6]

$$x = \varphi(q_1, \dots, q_n, \varepsilon^2), \quad x_n = q_n$$

we can reduce the kinetic-energy matrix to diagonal form. We have[6]

$$\varphi(q_1, \dots, q_{n-1}, 0, \varepsilon^2) = q, \;\; \frac{\partial \varphi}{\partial q_n} = -A^{-1}b \tag{2.21}$$

The kinetic energy takes the following form in the new coordinates

$$T = \frac{1}{2}(\hat{A}(q, q_n, \varepsilon^2)\dot{q}, \dot{q}) + \frac{1}{2}\hat{c}(q, q_n, \varepsilon^2)\dot{q}_n^2, \quad \hat{A} = (\hat{a}_{lk}), \quad l, k = 1, \dots, n-1 \tag{2.22}$$

$$\hat{a}_{lk} = \sum_{i, j = 1}^{n-1} a_{ij}\frac{\partial \varphi_i}{\partial q_l}\frac{\partial \varphi_j}{\partial q_k}\bigg|_{x, x_n \to q, q_n}, \quad \hat{c} = c - (A^{-1}b, b)\big|_{x, x_n \to q, q_n} \tag{2.23}$$

in which case $\hat{c}(q, 0, 0) = 0$, since the kinetic energy is degenerate on Λ.

We expand the potential energy and the coefficients occurring in the kinetic energy in series in powers of q_n

$$\hat{A}(q, q_n, 0) = \hat{A}_0(q) + \hat{A}_1(q)q_n + \frac{1}{2}\hat{A}_2(q)q_n^2 + O(q_n^3)$$

$$\hat{c}(q, q_n, 0) = \hat{c}_1(q)q_n + \frac{1}{2}\hat{c}_2(q)q_n^2 + O(q_n^3)$$

$$V(q, q_n, \varepsilon) = V_0(q, \varepsilon) + V_1(q, \varepsilon)q_n + \frac{1}{2}V_2(q, \varepsilon)q_n^2 + O(q_n^3)$$

It is obvious that the structure of system (2.7) does not change when $\hat{A}_1(q) = 0$, $\hat{c}_1(q) = 0$ and $V_1(q, \varepsilon) = \varepsilon^2\tilde{V}_1$. In view of this, the evenness conditions imposed on the potential and kinetic energies can be replaced by the condition that the derivative with respect to q_n is equal to zero when $q_n = 0$ and $\varepsilon = 0$, and the results obtained above remain true. For any function $f(x, x_n, \varepsilon)$, by the rule of the differentiation of a complex function and taking the replacement formula (2.21) into account, we have

$$\frac{\partial f}{\partial q_n}\bigg|_{q_n = 0, \varepsilon = 0} = \left(-\left(\frac{\partial f}{\partial x}\right)^T A^{-1}b + \frac{\partial f}{\partial x_n}\right)\bigg|_{x_n = 0, \varepsilon = 0}$$

By choosing as the function f the functions V and $\hat{c}$ of the form (2.23), we obtain the following conditions for Theorems 1 and 2 to be applicable

$$\left(\frac{\partial V}{\partial x}\right)^T A^{-1}b = \frac{\partial V}{\partial x_n}\bigg|_{x_n = 0, \varepsilon = 0}, \quad \left(\frac{\partial \hat{c}}{\partial x}\right)^T A^{-1}b = \frac{\partial \hat{c}}{\partial x_n}\bigg|_{x_n = 0, \varepsilon = 0} \tag{2.24}$$

We can similarly obtain the conditions for the coefficients of the kinetic-energy matrix A; these conditions are not given here in view of their complexity. If the kinetic energy of the system is independent of the variables x, we have

$\hat{a}_{lk} = a_{lk}|_{x_n \to q_n}$ and the conditions take the form

$$\left.\frac{\partial a_{lk}}{\partial x_n}\right|_{x_n = 0, \varepsilon = 0} = 0 \quad l, k = 1, \dots, n-1 \tag{2.25}$$

We will consider how to calculate the normal reaction of the constraint in the original variables. Taking the remark into account, we write the normal reaction (2.9) in the form

$$N(\dot{q}, q) = \left.\left(-\frac{1}{2}\left(\frac{\partial^2 \hat{A}}{\partial q_n^2}\dot{q}, \dot{q}\right) + \frac{\partial^2 V}{\partial q_n^2}\right)\right|_{q_n = 0, \varepsilon = 0} \tag{2.26}$$

Consequently, to calculate the normal reaction of the constraint it is necessary, in the same way as was done above, to calculate the second partial derivative of V and $\hat{A}$ with respect to q_n taking into account relations (2.21) and (2.23) and the fact that $\dot{q} = \dot{x}$ when $q_n = 0$. In the case when the function $f(x, x_n, \varepsilon)$ is independent of the variables x, we have

$$\partial^2 f/\partial q_n^2|_{q_n = 0, \varepsilon = 0} = \partial^2 f/\partial x_n^2|_{x_n = 0, \varepsilon = 0}$$

3. The problem of the motion of a double mathematical pendulum

Consider a system consisting of two point masses with masses m_1 and m_2, connected to one another by a weightless rod of length l_2, while the point with mass m_1 is connected to a fixed point by a weightless rod of length l_1 (Fig. 3). The system oscillates in a vertical plane. We will investigate the behaviour of the system in the case when the mass of the point closest to the suspension point is small: $m_1 = \varepsilon^2 \tilde{m}_1$. To be specific we will assume that $l_1 > l_2$. The case $l_1 < l_2$ is completely analogous. The case $l_1 = l_2$ is not covered by the proposed theory.

The system has two degrees of freedom. We will choose as the generalized coordinates the angles φ_1 and φ_2, which the rods make with the vertical. We will calculate the kinetic and potential energies of the system. The configuration space of the system is a two-dimensional torus. For the kinetic energy we have

$$T = \frac{1}{2}[(m_1 + m_2)l_1^2\dot{\varphi}_1^2 + 2m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1)\dot{\varphi}_1\dot{\varphi}_2 + m_2 l_2^2 \dot{\varphi}_2^2]$$

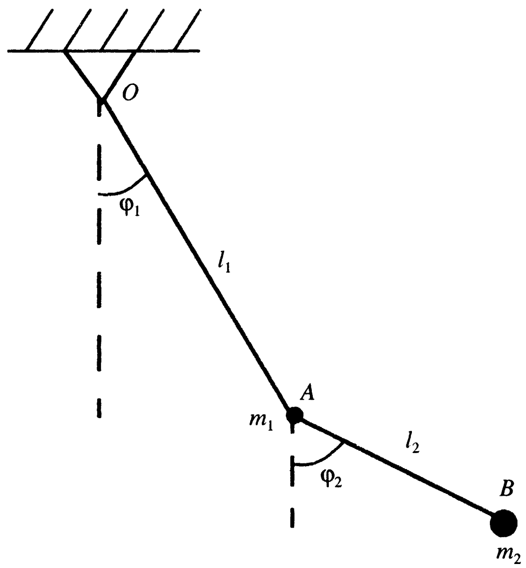

Fig. 3.

For the potential energy, if we assume it to be zero in the position $\varphi_1 = 0$ and $\varphi_2 = 0$, we will have

$$V = m_1 g l_1(1-\cos\varphi_1) + m_2 g(l_1(1-\cos\varphi_1) + l_2(1-\cos\varphi_2))$$

Assuming $m_1 = \varepsilon^2 \tilde{m}_1$, we can write the kinetic-energy matrix

$$A(\varphi_1, \varphi_2, \varepsilon^2) = \begin{Vmatrix} (m_2 + \varepsilon^2 \tilde{m}_1) l_1^2 & m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1) \\ m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1) & m_2 l_2^2 \end{Vmatrix} \tag{3.1}$$

The determinant of matrix (3.1) when $\varepsilon = 0$ is

$$\det A(\varphi_1, \varphi_2, 0) = m_2^2 l_1^2 l_2^2 \sin^2(\varphi_2 - \varphi_1)$$

Consequently, when $\varphi_2 = \varphi_1$ and $\varphi_2 = \varphi_1 + \pi$ the kinetic energy of the system is degenerate.

Hence, the curve of the degeneracy Λ splits into two components

$$\Lambda_1 = \{\varphi_2 = \varphi_1\}, \quad \Lambda_2 = \{\varphi_2 = \varphi_1 + \pi\}$$

Consider the behaviour of the system in the region of the component Λ_1. We make the replacement $\varphi_2 - \varphi_1 = \psi$. Then, the kinetic and potential energies take the form

$$T = \frac{1}{2} m_2 l_1^2 [a\dot{\varphi}_1^2 + 2b\dot{\varphi}_1\dot{\psi} + c\dot{\psi}^2] \tag{3.2}$$

$$V = V_0 - m_2 g l_1((\varepsilon^2 n + 1)\cos\varphi_1 + k\cos(\varphi_1 + \psi)) \tag{3.3}$$

where

$$a = 1 + 2k\cos\psi + k^2 + \varepsilon^2 n, \quad b = k(\cos\psi + k^2), \quad c = k^2$$

$$k = \frac{l_2}{l_1}, \quad n = \frac{\tilde{m}_1}{m_2}, \quad V_0 = m_2 g(l_1 + l_2) + \varepsilon^2 \tilde{m}_1 g l_1$$

According to what was said in Section 2.4, by a replacement of the variables of the form (2.21) we can reduce the kinetic-energy matrix to diagonal form. In this problem the variable φ_1 corresponds to the variable x while the variable ψ corresponds to the variable q_n. The replacement of the variables $(\varphi_1, \psi) \to (\theta, \psi)$ is given by the formula $\varphi_1 = f(\theta, \psi)$, where $f'_\psi = -b/a$.

Since the kinetic energy (3.2) of the system is independent of the variable φ_1, we obtain, as a result of the replacement,

$$T = \frac{1}{2} m_2 l_1^2 (a\dot{\theta}^2 + \hat{c}\dot{\psi}^2), \quad \hat{c} = \frac{ca - b^2}{a} = \frac{k^2(\sin^2\psi + \varepsilon^2 n)}{1 + 2k\cos\psi + k^2 + \varepsilon^2 n}$$

We will verify the conditions required for Theorems 1 and 2 to be satisfied. For the potential energy (3.3) we have $\partial V/\partial\psi|_{\psi=0,\, \varepsilon=0} = 0$. By virtue of the fact that a and $\hat{c}$ are even in ψ, conditions (2.24) and (2.25), for which Theorems 1 and 2 are applicable, are satisfied.

We will obtain the normal reaction of the constraint $\psi = 0$. By formula (2.26) we have

$$N(\dot{\theta}, \theta) = \left. \left(-\frac{1}{2} m_2 l_1^2 \frac{\partial^2 a}{\partial \psi^2} \dot{\theta}^2 + \frac{\partial^2 V}{\partial \psi^2} \right) \right|_{\psi = 0,\, \varepsilon = 0}$$

Taking into account the fact that $(\cos\psi)'|_{\psi=0} = 0$ and $\varphi_1 = 0$ when $\psi = 0$, we obtain

$$\left. \frac{\partial^2 V}{\partial \psi^2} \right|_{\psi = 0,\, \varepsilon = 0} = \frac{m_2 l_1 g k}{1+k} \cos\theta, \quad \frac{\partial^2 a}{\partial \psi^2} = -2k\cos\psi$$

Then

$$N(\dot{\theta}, \theta) = \frac{m_2 l_1 k}{k+1}[(l_1 + l_2)\dot{\theta}^2 + g\cos\theta] \tag{3.4}$$

As was mentioned above (see the remark), the function (3.4) is not exactly the physical normal reaction of the constraint, but is proportional to it.

For motion with respect to the constraint, $\dot{\theta}$ and θ satisfy the equality

$$\frac{1}{2}m_2(l_1 + l_2)^2\dot{\theta}^2 + m_2 g(l_1 + l_2)(1 - \cos\theta) = h \tag{3.5}$$

It can be seen from relations (3.4) and (3.5) that when $\psi = 0$ the behaviour of the double pendulum is described by the motion of a point with mass m_2 in a circle of radius $R = l_1 + l_2$. By expressing $\dot{\theta}$ from Eq. (3.5) and substituting it into (3.4), we obtain that

$$N = \frac{m_2 l_1 k}{1+k}(C - 2g + 3g\cos\theta); \quad C = \frac{2h}{m_2(l_1 + l_2)}$$

Consequently, the normal reaction is positive if $\cos\theta > 2/3 - C/(3g)$. If

$$h > \frac{5}{2}m_2(l_1 + l_2)g \tag{3.6}$$

the normal reaction of the component of the boundary Λ_1 is positive everywhere, and so the trajectories of the limit system, beginning at the boundary Λ_1 with velocity tangent to Λ_1 do not leave the boundary. If the sign of inequality (3.6) is reversed, then, on a certain part of the component of the boundary Λ_1, specified by the angles $\theta \in (-\theta^*, \theta^*)$, where $\theta^* = \arccos(2/3 - C/(3g))$, the normal reaction is positive, while on the part of the boundary Λ_1 specified by the angles $\theta \in (\theta^*, 2\pi - \theta^*)$, the normal reaction is negative.

We can similarly consider the motion in the region of the component Λ_2. The equation of the constraint in this case has the form $\psi = \pi$, and a normal reaction of the constraint is

$$N(\dot{\theta}, \theta) = -\frac{m_2 l_1 k}{1-k}[(l_1 - l_2)\dot{\theta}^2 + g\cos\theta]$$

For motion on the constraint, $\dot{\theta}$ and θ satisfy the equality

$$\frac{1}{2}m_2(l_1 - l_2)^2\dot{\theta}^2 + m_2 g(l_1 - l_2)(1 - \cos\theta) = h - 2m_2 g l_2$$

It can be seen that the behaviour of the double pendulum when $\psi = \pi$ is described by the motion of a point of mass m_2 on the outer side of a circle of radius $r = l_1 - l_2$. As in the previous case, a region of the component of the boundary Λ_2 exists where the normal reaction is negative, and a region of the component of the boundary Λ_2 where the normal reaction is positive. For sufficiently large h the normal reaction is negative everywhere on the boundary $\psi = \pi$.

Hence, when $\varepsilon \to 0$ for a fixed level of energy of the system h, the motions of the limit system are the motions of a point of mass m_2 in a gravitational field inside a ring

$$D = \{z = (x, y): r \le |z| \le R\}; \quad r = l_1 - l_2, \quad R = l_1 + l_2$$

On the boundaries of the ring the kinetic energy is degenerate. If the trajectories of the limit system reach the boundary at a non-zero angle, the behaviour of the trajectories are such that the angle of incidence is equal to the angle of reflection (billiard trajectories).[4]

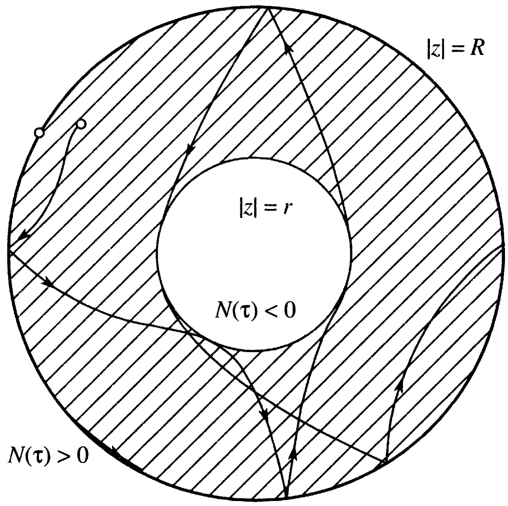

Fig. 4.

Other types of trajectories exist in the neighbourhood of the boundary in the limit system. These are:

1) if the normal reaction is positive on a certain part of the boundary, the trajectories beginning in this part with a velocity tangential to the boundary proceed along the boundary until they arrive in the region of negative reaction;
2) if the normal reaction of the boundary is negative on a certain part of the boundary, then almost all the trajectories leave the boundary, but a single boundary exists which connects any two points of the boundary and which passes along it (Fig. 4).

When the energy h of the system is sufficiently high, the normal reaction of the constraint $|z| = R$ is positive at each point of the boundary, while the normal reaction of the constraint $|z| = r$ is negative.

The behaviour of a double mathematical pendulum in the case when the mass of the point closest to the suspension point is small, was considered previously in Ref. 10. For values of the energy close to the maximum potential energy, using variational methods it was proved that the problem of a double mathematical pendulum is non-integrable for a certain ratio of the masses of the point masses and of the lengths of its sections. The existence of chaotic trajectories of a double pendulum and the non-integrability of the system for sufficiently small ratios of the masses and sufficiently high energy values were proved in Ref. 11 using methods of perturbation theory.

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